Measures on Operational Logics *

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It is demonstrated, that any measure on effects is generated by a state, whereas a similar statement for operations is not true.

The operational approach to quantum mechanics [1, 2, 3] introduces at least two sets which to some extent could be regarded as generalizations of the "quantum logic": the set of effects (also simple observables, tests), and the set of operations. Both sets carry natural structures making it possible to define generalized measures on them. Moreover, any state induces such measures, hence the question arises: is any measure on effects (resp. operations) generated by some state? This problem, formulated by R. S. Ingarden ([4], p. 115) is solved below.

We start with three postulates [5] which underlie the operational approach to quantum mechanics.

Postulate 1. The set of states of a physical system is represented by the set of elements of a closed generating cone S for a complete base norm space (V, S_1).

Postulate 2. The set L_{ef} of effects is represented by the unit interval $\langle o, e \rangle$ in the dual space (V^*, e) of (V, S_1).

Postulate 3. The set L_{op} of operations on the physical system is represented by the set of positive elements in the unit ball of $\mathscr{L}(V)$.

We define bounded measures on L_{ef} [4] as mappings $\mu: L_{ef} \mapsto \mathbb{R}$ such that:

(i) $\mu(a) \geqq 0$ for any $a \in L_{ef}$,

(ii) $\mu(0) = 0, \quad \mu(e) < \infty$,

(iii) $\sum_{k=1}^{\infty} a_k \in L_{ef} \mid\Rightarrow \mu\left(\sum_{k=1}^{\infty} a_k\right) = \sum_{k=1}^{\infty} \mu(a_k)$

for any sequence of effects with the sum converging in the weak* topology.

Any state (i.e. any element of the cone S) generates a weak* continuous positive linear functional on V^*, which when restricted to L_{ef} defines a measure in the above sense. It is evident, however, that it suffices to take a sequentially weak* continuous positive functional on V^* to obtain a measure on L_{ef}. So our problem reduces to the question: are there positive linear sequentially weak* continuous functionals on V^* which are not weak* continuous?

Let α be such a functional, then $\alpha^{-1}(0)$ is a sequentially weak* closed, but weak* dense, hypersubspace of V^*. As the positive cone S^* of V^* has non-empty weak* interior, then $\alpha^{-1}(0) \cap S^*$ cannot be empty. It suffices for the conclusion: any positive linear sequentially weak* continuous functional on V^* is also weak* continuous, hence any measure on L_{ef} is generated by a state.

One defines measures on L_{op} [3, 4] as mappings $\mu: L_{op} \mapsto \mathbb{R}$ such that:

(i) $\mu(\Phi) \geqq 0$ for any $\Phi \in L_{op}$,

(ii) $\mu(\Omega) = 0, \quad \mu(I) < \infty$,

(iii) $\sum_{k=1}^{\infty} \Phi_k \in L_{op} \mid\Rightarrow \mu\left(\sum_{k=1}^{\infty} \Phi_k\right) = \sum_{k=1}^{\infty} \mu(\Phi_k)$

for any sequence of operations with the sum converging in the strong operator topology, where $\Omega\alpha = \omega$ (the zero-vector of V) for any $\alpha \in V$, $I\alpha = \alpha$ for any $\alpha \in V$.

Any linear positive sequentially strong continuous functional on $\mathscr{L}(V)$ generates measure on L_{op}, and conversely. States generate standard weak continuous functionals $\Phi \mapsto e(\Phi\alpha)$, $\alpha \in S$. Observe that these functionals do not separate elements of $\mathscr{L}(V)$. As the family $\{a(\cdot\,\alpha) \mid a \in V^*, \alpha \in S, a \geqq 0\}$ of positive weak continuous functionals on $\mathscr{L}(V)$ does separate elements of $\mathscr{L}(V)$, we conclude that the set of measures on L_{op} is far larger than the set of state generated ones.

In the standard von Neumann model of quantum mechanics we start with a von Neumann algebra $\mathscr{B}$ on a Hilbert space. The space (V, S_1) is then realized as the space of all self-adjoint ultra-weak continuous linear functionals on $\mathscr{B}$, with S_1 — the convex set of normalized normal positive functionals on $\mathscr{B}$. The strictly positive linear functional e on V, corresponding to the base S_1, is just the trace. V^* can be identified with the set of self-adjoint elements of $\mathscr{B}$. It is rather obvious now that in this model any measure on L_{ef} is generated

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by a state (is regular, in terms of Ingarden). This fact could be regarded as a generalization of the Gleason theorem. Our general conclusion concerning measures on operations holds also in this case.

It would be of interest to study states on a special class of operations, called filters (or strongly repeatable operations), as well on the corresponding effects, called decision effects. This is not an easy task, however, as there is no agreement about the precise notion of filter (compare definitions in [2], p. 200, [1], p. 254, and a discussion in [6], where further references can be found).

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Berichtigung

Chr. Bräuchle, H. Kabza, and J. Voitländer, Optical and ODMR Investigations of the Lowest Excited Triplet State of Dinaphtho-(2'.3':1.2); (2''.3'':6.7)-pyrene, Z. Naturforsch. **34a**, 6 [1979].

In dieser Arbeit müssen die Fußnoten [6] und [7] wie folgt lauten:

[6] Chr. Bräuchle, H. Kabza, J. Voitländer, and E. Clar, Chem. Phys. **32**, 63 (1978).
[7] E. Clar, W. Schmidt, and T. Zauner, to be published.

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